

Monotonicity of Subelliptic Estimates on Rigid Pseudoconvex Domains

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Abstract

In this paper we will present monotonicity of subelliptic estimates involving Levi forms on rigid pseudoconvex domains. As an application of monotonicity, we will show that if a rigid domain is given by a sum of the squares of monomials and if the domain is of finite type, then the sharp subelliptic estimate of this domain equals the reciprocal of the D'Angelo's 1-type.

1 Introduction

This paper mainly concerns the following class of domains. Let Ω be a pseudoconvex domain in \mathbb{C}^{n+1} and let the origin be a boundary point of Ω . Following the terminology in [BRT], we say that Ω is a *rigid pseudoconvex domain* near the origin if there exists a complex coordinate (z_1, \dots, z_{n+1}) such that Ω is defined near the origin by

$$\Omega = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : \operatorname{Re} z_{n+1} + u(z_1, \dots, z_n) < 0\}, \quad (1)$$

where u is a plurisubharmonic function depending only on the first n -variables with $u(0) = 0$. We say that u is the *mixed term* of the boundary defining function of Ω near the origin.

Let Ω be a smoothly bounded pseudoconvex domain in \mathbb{C}^{n+1} and let p_0 be a point on the boundary of Ω . We say that *subelliptic estimate of order $\epsilon > 0$* holds on $(0, 1)$ -forms at p_0 if there exist a constant C and a neighborhood U of p_0 such that the estimate

$$|||\varphi|||_\epsilon^2 \leq C(||\bar{\partial}\varphi||^2 + ||\bar{\partial}^*\varphi||^2 + ||\varphi||^2) \quad (2)$$

is valid for all $\varphi \in \mathcal{D}^{0,1}(U)$. Here $|||\cdot|||_\epsilon$ denotes the tangential Sobolev norm of order ϵ and $\mathcal{D}^{0,1}(U)$ refers to the set of smooth $(0, 1)$ -forms φ in $\text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*)$. The *D'Angelo's 1-type* of the boundary of Ω at p_0 is defined by

$$T(b\Omega, p_0) = \sup_z \frac{v(z^*r)}{v(z)}. \quad (3)$$

Here z runs over the set of all germs of parameterized complex analytic curves at p_0 and $v(\cdot)$ denotes the vanishing order. Even though the type function, $p \rightarrow T(b\Omega, p)$, is neither upper nor lower semi-continuous, D'Angelo showed that if $T(b\Omega, p_0)$ is finite, then it is locally bounded near p_0 [DA82].

Catlin showed that if Ω is a smoothly bounded pseudoconvex domain in \mathbb{C}^{n+1} and $p_0 \in b\Omega$, then $T(b\Omega, p_0)$ is finite if and only if there is a subelliptic estimate of some order $\epsilon > 0$ on $(0, 1)$ -forms at p_0 . More precisely, for necessary condition of subellipticity he [Ca83] showed that if $T(b\Omega, p_0)$ is finite and a subelliptic estimate of order ϵ of the form (2) holds, then ϵ must satisfy $\epsilon \leq \frac{1}{T(b\Omega, p_0)}$. For sufficiency he reduced the problem to construct a family of plurisubharmonic functions $\{\lambda_\delta\}$ with large Hessians near the boundary.

Theorem (Catlin [Ca87, Theorem 2.2]). *Let p_0 be a point on the boundary of a smoothly bounded pseudoconvex domain in \mathbb{C}^{n+1} . Let $\Omega_\delta = \{z \mid r(z) < \delta\}$ and let $S_\delta = \{z \mid -\delta < r(z) < \delta\}$. Suppose that there exist a neighborhood U of p_0 , a parameter ϵ with $0 < \epsilon \leq \frac{1}{2}$, and a constant $c > 0$ such that for any sufficiently small $\delta > 0$ there exists a smooth plurisubharmonic function λ_δ in $\tilde{U} \cap \Omega_\delta$ satisfying the following properties:*

$$(i) \quad |\lambda_\delta| \leq 1 \quad \text{on } \tilde{U} \cap \Omega_\delta \quad (4)$$

$$(ii) \quad \sum_{i,j=1}^{n+1} \frac{\partial^2 \lambda_\delta}{\partial z_i \partial \bar{z}_j} s_i \bar{s}_j \geq c\delta^{-2\epsilon} \sum_{i=1}^{n+1} |s_i|^2, \quad s_i \in \mathbb{C}, \quad \text{on } \tilde{U} \cap S_\delta. \quad (5)$$

Then there exists a neighborhood \tilde{U}' of p_0 with $\tilde{U}' \subset \subset \tilde{U}$ such that a subelliptic estimate of order ϵ holds in \tilde{U}' .

In this paper we are primarily interested in a construction of a family of plurisubharmonic functions, $\{\lambda_\delta\}$, satisfying (4) and (5), for a rigid pseudoconvex domain defined by (1). The main idea of this paper is to resolve $\partial\bar{\partial}u$ at degenerated points by adding small cutoff functions in a uniform way. Since the mixed term u is independent of the last variable z_{n+1} , there is a natural projection from the boundary of a rigid domain to a neighborhood of the origin in \mathbb{C}^n . Under this projection one can interpret the Levi form of the boundary of Ω in terms of $\partial\bar{\partial}u$. This observation enables us to derive the following lemma, a modified version of [Ca87, Theorem 2.2] for rigid pseudoconvex domains.

Main Lemma. *Let $u(z)$ be a smooth plurisubharmonic function in a neighborhood V of the origin in \mathbb{C}^n with $u(0) = 0$ and let Ω be a rigid pseudoconvex domain defined by (1) near the origin in \mathbb{C}^{n+1} . Suppose that there exist a neighborhood $U \subset\subset V$ of the origin in \mathbb{C}^n , a constant $C > 0$, and a parameter ϵ with $0 < \epsilon \leq \frac{1}{2}$ so that for each sufficiently small $\delta > 0$ there exists a smooth function $\rho_\delta(z)$ on U with the following properties:*

(i) $0 \leq \rho_\delta(z) \leq 1$ for all $z \in U$,

(ii) for all $z \in U$ and for all $L = s_1 \frac{\partial}{\partial z_1} + \cdots + s_n \frac{\partial}{\partial z_n}$, $s_i \in \mathbb{C}$

$$\partial\bar{\partial} \left(\frac{u}{\delta} + \rho_\delta \right) (L, \bar{L})(z) \geq C\delta^{-2\epsilon} |L|^2. \quad (6)$$

Then a subelliptic estimate of order ϵ holds at the origin for Ω .

We will give the proof of Main Lemma in Section 2. As an immediate consequence of Main Lemma, we obtain the following monotone property of subelliptic estimates on two rigid domains.

Theorem 1. *Let u_1 and u_2 be plurisubharmonic functions defined on a neighborhood V of the origin in \mathbb{C}^n with $u_1(0) = u_2(0) = 0$. Let Ω_1 and Ω_2 be rigid pseudoconvex domains with mixed terms, u_1 and u_2 , respectively, near the origin in \mathbb{C}^{n+1} . Suppose that there exist a neighborhood $U \subset\subset V$ in \mathbb{C}^n , a constant $C > 0$, and a parameter ϵ with $0 < \epsilon \leq \frac{1}{2}$ such that for each sufficiently small $\delta > 0$ there exists a smooth function $\rho_\delta(z)$ satisfying (i) and (ii) for u_2 in Main Lemma. If for all $z \in V$,*

$$\partial\bar{\partial}u_1(L, \bar{L})(z) \geq \partial\bar{\partial}u_2(L, \bar{L})(z), \quad L = \sum_{i=1}^n s_i \frac{\partial}{\partial z_i}, \quad (7)$$

then a subelliptic estimate of order ϵ holds at the origin for both Ω_1 and Ω_2 .

In Section 3, as an application of the monotone property, we will consider the largest subelliptic gain of a rigid pseudoconvex domain whose Levi form is greater than or equal to the one of a diagonal domain (Theorem 2). As a corollary of Theorem 2, We will provide an answer to the D'Angelo's conjecture when the mixed term of a rigid pseudoconvex domain is a sum of the squares of monomials (Corollary 2).

Conjecture (D'Angelo [DA93]). Let \mathcal{O}_n be the ring of germs of holomorphic functions at the origin \mathbb{C}^n and let I be an ideal generated by $f_1, \dots, f_l \in \mathcal{O}_n$ with $f_j(0) = 0$ $j = 1, \dots, l$. Let $m(I)$ denote $\dim_{\mathbb{C}} \mathcal{O}_n/I$. Let $\Omega \subset \subset \mathbb{C}^{n+1}$ be a rigid domain whose boundary near the origin is defined by $r(z') = 2 \operatorname{Re} z_{n+1} + \sum_{i=1}^l |f_i(z)|^2$, where $z = (z_1, \dots, z_n)$ and $z' = (z_1, \dots, z_{n+1})$. If $m(I)$ is finite, then a subelliptic estimate holds for

$$\frac{1}{2m(I)} \leq \epsilon \leq \frac{1}{T(b\Omega, 0)}. \quad (8)$$

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2 Proof of Main Lemma

Let $\Omega \subset \mathbb{C}^{n+1}$ be a rigid domain whose boundary is defined near the origin by

$$r(z') = 2 \operatorname{Re} z_{n+1} + u(z), \quad (9)$$

where $u(z)$ is a smooth plurisubharmonic function, depending only on $z = (z_1, \dots, z_n)$, with $u(0) = 0$. Let \tilde{U} be an open set in \mathbb{C}^{n+1} defined by $\tilde{U} = U \times \mathbb{C} \subset \mathbb{C}^{n+1}$, and let

$$\Omega_\delta = \{z' \in \mathbb{C}^{n+1} \mid r(z') < \delta\} \quad (10)$$

$$S_\delta = \{z' \in \mathbb{C}^{n+1} \mid -\delta < r(z') \leq 0\}. \quad (11)$$

For simplicity, we set

$$\partial_i = \frac{\partial}{\partial z_i} \quad i = 1, \dots, n+1. \quad (12)$$

Let denote a constant vector field in \mathbb{C}^n by

$$L = \sum_{i=1}^n s_i \partial_i \quad s_i \in \mathbb{C}. \quad (13)$$

Consider $(1, 0)$ -vector fields L_1, \dots, L_{n+1} on \tilde{U} , defined by

$$\begin{aligned} L_i &= \partial_i - \frac{r_{z_i}}{r_{z_{n+1}}} \partial_{n+1} = \partial_i - u_{z_i} \partial_{n+1}, \quad i = 1, \dots, n \\ L_{n+1} &= \partial_{n+1}. \end{aligned} \quad (14)$$

Note that $L_i r = 0$ with $i = 1, \dots, n$ for all $z' \in \tilde{U}$. Let us denote

$$L' = \sum_{i=1}^{n+1} s_i L_i, \quad \tilde{L} = \sum_{i=1}^{n+1} s_i \partial_i, \quad s_i \in \mathbb{C}.$$

Since u_{z_i} is bounded on $U \subset\subset V$ for $i = 1, \dots, n$, it follows that there exist constants $C_1, C_2 > 0$, depending only on U , so that

$$C_1 |L'|^2 \leq \sum_{i=1}^{n+1} |s_i|^2 \leq C_2 |\tilde{L}|^2. \quad (15)$$

Step 1. Let $\tilde{\lambda}_\delta$ be a function defined on $V \times \mathbb{C} \subset \mathbb{C}^{n+1}$ by

$$\tilde{\lambda}_\delta = e^{\frac{r}{\delta}} + e^{-3} \rho_\delta. \quad (16)$$

If $z' \in U \times \mathbb{C}$ with $r(z') \geq -3\delta$, then for all L' at z' ,

$$\partial \bar{\partial} \tilde{\lambda}_\delta(L', \bar{L}')(z') \gtrsim \delta^{-2\epsilon} |L'|^2. \quad (17)$$

Proof of Step 1. Note that

$$\partial \bar{\partial}(\tilde{\lambda}_\delta) = e^{\frac{r}{\delta}} \left[\frac{\partial r \wedge \bar{\partial} r}{\delta^2} + \frac{\partial \bar{\partial} u}{\delta} \right] + \frac{\partial \bar{\partial} \rho_\delta}{e^3}. \quad (18)$$

Since

$$\begin{aligned} \partial \bar{\partial} u(L', \bar{L}')(z') &= \partial \bar{\partial} u(L, \bar{L})(z) \\ \partial \bar{\partial} \rho_\delta(L', \bar{L}')(z') &= \partial \bar{\partial} \rho_\delta(L, \bar{L})(z), \end{aligned} \quad (19)$$

it follows that for any $z' = (z, z_{n+1}) \in \tilde{U}$ with $r(z') \geq -3\delta$,

$$\begin{aligned}
\partial\bar{\partial}\tilde{\lambda}_\delta(L', \bar{L}')(z') &= e^{\frac{r}{\delta}} \left[\frac{(\partial r \wedge \bar{\partial} r)}{\delta^2}(L', \bar{L}')(z') \right] \\
&\quad + e^{\frac{r}{\delta}} \left[\frac{\partial\bar{\partial}u}{\delta}(L, \bar{L})(z) \right] + \frac{\partial\bar{\partial}\rho_\delta}{e^3}(L, \bar{L})(z) \\
&= e^{\frac{r}{\delta}} \left[\frac{|s_{n+1}|^2}{\delta^2} \right] + e^{\frac{r}{\delta}} \left[\frac{\partial\bar{\partial}u}{\delta}(L, \bar{L})(z) \right] + \frac{\partial\bar{\partial}\rho_\delta}{e^3}(L, \bar{L})(z) \\
&\geq e^{-3} \left[\frac{|s_{n+1}|^2}{\delta^2} + \partial\bar{\partial} \left(\frac{u}{\delta} + \rho_\delta \right) (L, \bar{L})(z) \right] \\
&\geq e^{-3} \left[\frac{|s_{n+1}|^2}{\delta^2} + C\delta^{-2\epsilon}|L|^2 \right]. \tag{20}
\end{aligned}$$

In fact, the first equality follows from (18) and (19). The second equality results from the fact that $L_i r = 0$ for $1 \leq i \leq n$ and the third inequality is obtained by the fact that $e^{\frac{r(z')}{\delta}} \geq e^{-3}$ for $r(z') \geq -3\delta$. The fourth inequality follows from (6). Since $0 < \epsilon \leq \frac{1}{2}$ and δ is small, it follows from (15) and (20) that (17) holds. \square

Step 2. Let $p(t)$ be an increasing smooth convex function with $p(t) = 0$ for $t \leq e^{-2}$, and $p(t) > 0$, $p'(t) > 0$, for $t > e^{-2}$. Let

$$\lambda_\delta(z') = p \circ \tilde{\lambda}_\delta(z'). \tag{21}$$

Then λ_δ is a plurisubharmonic function on \tilde{U} .

Proof of Step 2. Since $\tilde{\lambda}_\delta = e^{\frac{r}{\delta}} + e^{-3}\rho_\delta$ and $0 \leq \rho_\delta \leq 1$, it follows that

$$\tilde{\lambda}_\delta(z') \leq e^{-3} + e^{-3} < e^{-2}, \quad \text{for } z' \in \tilde{U} \text{ with } r(z') \leq -3\delta. \tag{22}$$

Since $p(t) = 0$ for $t \leq e^{-2}$, we have

$$\lambda_\delta(z') = p(\tilde{\lambda}_\delta(z')) = 0, \quad \text{for } z' \in \tilde{U} \text{ with } r(z') \leq -3\delta. \tag{23}$$

If $z' \in \tilde{U}$ with $r(z') \geq -3\delta$, then since $\tilde{\lambda}_\delta(z')$ is plurisubharmonic from Step 1 and since $p(t)$ is a convex increasing function, it follows that λ_δ is plurisubharmonic on $z' \in \tilde{U}$ with $r(z') \geq -3\delta$. Therefore, combining with (23), we conclude that λ_δ is plurisubharmonic on \tilde{U} . \square

Continue the proof of Main Lemma 1. Since $\tilde{\lambda}_\delta(z') \leq e + e^{-3}$ for $z' \in \tilde{U} \cap \Omega_\delta$ and since $\lambda(z') = p \circ \tilde{\lambda}_\delta(z') = 0$ for $z' \in \tilde{U}$ with $r(z') < -3\delta$ by (23), it follows that there exists a constant $C' > 0$ independent of δ such that

$$|\lambda_\delta(z')| \leq C', \quad \text{for all } z' \in \tilde{U} \cap \Omega_\delta. \quad (24)$$

Since p is a smooth convex increasing function with $p'(t) > 0$ for $t > e^{-2}$, there exists a constant $c > 0$ so that

$$p'(t) \geq c, \quad \text{for } e^{-1} \leq t \leq e + e^{-3}. \quad (25)$$

Since

$$e^{-1} \leq \tilde{\lambda}_\delta(z') \leq e + e^{-3} \quad \text{for } z' \in \tilde{U} \cap S_\delta, \quad (26)$$

it follows from (17) and (25) that there exists a constant $C'' > 0$ independent of δ such that for all $z' \in S_\delta \cap \tilde{U}$ and all L' at z' ,

$$\partial\bar{\partial}\lambda_\delta(L', \bar{L}')(z') \geq C''\delta^{-2\epsilon}|L'|^2.$$

3 Sharp Subelliptic Estimates on Rigid Monomial Domains

In this section we will consider subellipticity of a rigid domain whose Levi form is greater than or equal to the one of a diagonal domain. As an application of monotonicity, we will show that if the mixed term of the boundary of a rigid domain is given by sum of the squares of monomials and it is of finite type at the origin, then the sharp subelliptic estimate of this domain is the reciprocal of the 1-type.

Theorem 2. *Let Ω be a rigid pseudoconvex domain near the origin and let u be the mixed term of the boundary of Ω near the origin. Let m_1, \dots, m_n be positive integer. Suppose that there exists a neighborhood V of the origin in \mathbb{C}^n so that for all $z \in V$*

$$\partial\bar{\partial}u(L, \bar{L})(z) \geq \partial\bar{\partial}\left(\sum_{i=1}^n |z_i|^{2m_i}\right)(L, \bar{L}) \quad \text{for } L = \sum_{i=1}^n s_i \partial_i. \quad (27)$$

Then a subelliptic estimate at the origin for Ω holds of order

$$\epsilon = \frac{1}{2 \max\{m_i : 1 \leq i \leq n\}}. \quad (28)$$

Proof. By Theorem 1 it is enough to show that there exist a neighborhood U of the origin in \mathbb{C}^n , a constant $C > 0$, and a parameter ϵ with $0 < \epsilon \leq \frac{1}{2}$ such that for each sufficiently small $\delta > 0$ there exists a smooth function ρ_δ satisfying (i) and (ii) for $\sum_{i=1}^n |z_i|^{2m_i}$ in Main Lemma.

Let χ be a smooth function such that $0 \leq \chi(t) \leq 1$ for $t \geq 0$, $\chi(t) = t$ for $t \leq \frac{1}{2}$, and $\chi(t) = 0$ for $t \geq 1$. Let us denote $m = \max\{m_i : 1 \leq i \leq n\}$. For each small $\delta > 0$ define $\tau_i(\delta) = \delta^{\frac{1}{2m_i}}$, $1 \leq i \leq n$ and

$$\rho_\delta(z) = c \sum_{i=1}^n \chi\left(\frac{|z_i|^2}{(\tau_i(\delta))^2}\right), \quad (29)$$

where $c > 0$ is to be determined. Let $U \subset\subset V$ be neighborhood of the origin. For $L = \sum_{i=1}^n s_i \partial_i$ and $z \in U$,

$$\begin{aligned} & \partial \bar{\partial} \left(\frac{\sum_{i=1}^n |z_i|^{2m_i}}{\delta} + \rho_\delta \right) (L, \bar{L})(z) \\ &= \partial \bar{\partial} \left(\frac{\sum_{i=1}^n |z_i|^{2m_i}}{\delta} + c \sum_{i=1}^n \chi\left(\frac{|z_i|^2}{(\tau_i(\delta))^2}\right) \right) (L, \bar{L})(z) \\ &= \sum_{i=1}^n a_i |s_i|^2, \end{aligned} \quad (30)$$

where for $i = 1, \dots, n$,

$$a_i = \frac{|z_i|^{2m_i-2}}{\delta} + c \chi' \left(\frac{|z_i|^2}{(\tau_i(\delta))^2} \right) \frac{1}{(\tau_i(\delta))^2} + c \chi'' \left(\frac{|z_i|^2}{(\tau_i(\delta))^2} \right) \frac{|z_i|^2}{(\tau_i(\delta))^4}. \quad (31)$$

Since a_i depends only on the i -th variable z_i , it is enough to estimate the constant a_i for each variable z_i . If $|z_i|^2 \geq (\tau_i(\delta))^2$, then since the last two terms in (31) vanish, it follows that

$$a_i \geq \frac{(\tau_i(\delta))^{2m_i-2}}{\delta} = \delta^{-\frac{1}{m_i}}, \quad 1 \leq i \leq n. \quad (32)$$

If $|z_i|^2 \leq \frac{1}{2}(\tau_i(\delta))^2$, then since $\chi'(\frac{|z_i|^2}{(\tau_i(\delta))^2}) = 1$ and $\chi''(\frac{|z_i|^2}{(\tau_i(\delta))^2}) = 0$, we have

$$a_i \geq c \frac{1}{(\tau_i(\delta))^2} = c \delta^{-\frac{1}{m_i}}, \quad 1 \leq i \leq n. \quad (33)$$

Let us denote

$$M = \sup\{|\chi'(t)| + |\chi''(t)| : 0 \leq t \leq 1\}. \quad (34)$$

If $\frac{1}{2}(\tau_i(\delta))^2 \leq |z_i|^2 \leq (\tau_i(\delta))^2$, then

$$a_i \geq (2^{1-m_i} - cM)\delta^{-\frac{1}{m_i}}. \quad (35)$$

Since m_i and M are independent of δ , we can choose a small number $c > 0$, independent of δ , to make $2^{1-m_i} - cM$ to be positive for all $i = 1, \dots, n$. Letting $d = \min_{1 \leq i \leq n} \{2^{1-m_i} - cM\}$, we have

$$a_i \geq d\delta^{-\frac{1}{m_i}}, \quad 1 \leq i \leq n. \quad (36)$$

Combining (32), (33), and (36), we conclude that there exists a constant $C > 0$, independent of δ , so that $a_i \geq C\delta^{-\frac{1}{m_i}}$, $1 \leq i \leq n$, for each small δ . Hence, we complete the proof. \square

Corollary 1. *Let $v(z)$ be a plurisubharmonic function near the origin in \mathbb{C}^n and let $u(z) = v(z) + \sum_{i=1}^n |z_i|^{2m_i}$, where m_i 's are positive integers. If Ω be a rigid domain with the mixed term $u(z)$ near origin in \mathbb{C}^{n+1} , then a subelliptic estimate holds at the origin of order $\epsilon = \frac{1}{2m}$, where $m = \max_{1 \leq i \leq n} m_i$.*

Corollary 2. *Let f_1, \dots, f_l be monomials in $z = (z_1, \dots, z_n)$ and let Ω be a pseudoconvex domain in \mathbb{C}^{n+1} whose boundary defining function near the origin is*

$$2 \operatorname{Re}(z_{n+1}) + \sum_{j=1}^l |f_j(z)|^2. \quad (37)$$

If $T(b\Omega, 0) < \infty$, then the best subelliptic estimate at the origin for Ω is

$$\epsilon = \frac{1}{T(b\Omega, 0)}, \quad (38)$$

Proof. Since $T(b\Omega, 0) < \infty$ and f_1, \dots, f_l are monomials, it follows that some positive power of each coordinate z_i , $i = 1, \dots, n$, equals one of f_1, \dots, f_l up to constant. Let m_i be the smallest one among such powers of z_i . Then, the mixed term of the boundary defining function near the origin satisfies (27). Furthermore, one can show that $T(b\Omega, 0) = \max\{m_i \mid i = 1, \dots, n\}$. Therefore, Theorem 2 and [Ca83, Theorem 3] implies the corollary. \square

References

- [BRT] M. S. Baouendi, L. Rothschild, and F. Treves, *CR structures with group action and extendability of CR functions*, Invent. Math. 82 (1985), no. 2, 359–396
- [Ca83] D. Catlin, *Necessary conditions for subellipticity of the $\bar{\partial}$ -Neumann problem*, Ann. of Math. (2) 117 (1983), 147-171.
- [Ca84] D. Catlin, *Boundary invariants of pseudoconvex domains*, Ann. of Math. (2) 120 (1984), 529-586.
- [Ca87] D. Catlin, *Subelliptic estimates for the $\bar{\partial}$ -Neumann problem on pseudoconvex domains*, Ann. of Math. (2) 126 (1987), 131-191.
- [Ca89] D. Catlin, *Estimates of invariant metrics on pseudoconvex domains of dimension two*, Math. Z. 200 (1989), 429-466.
- [DA82] J.P. D’Angelo, *Real hypersurfaces, order of contact, and applications*, Ann. of Math. (2) 115:3 (1982), 615-637.
- [DA93] J.P. D’Angelo, *Several complex variables and geometry of real hypersurfaces*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1993.
- [DA08] J.P. D’Angelo, *Kohn’s algorithm for subelliptic multipliers*, to appear as Lecture 6 in Real and Complex Geometry meet the Cauchy-Riemann Equation, Park City Math Institute Lecture Notes, 2008.
- [DF] K. Diederich and J. E. Fornaess, *Pseudoconvex domains with real-analytic boundary*, Ann. of Math. 107 (1978), 371-384.
- [FK] G. Folland and J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Annals of Math. Studies 75, 1972, Princeton Univ. Press.
- [G] P. Greiner, *On subelliptic estimates of the $\bar{\partial}$ -Neumann problem in \mathbb{C}^2* , J. Diff. Geom. 9 (1974), 239-250.
- [H] A.-K. Herbig, *A sufficient condition for subellipticity of the $\bar{\partial}$ -Neumann operator*, J. of Func. Anal. 242 (2007), 337-362.

- [K72] J.J. Kohn, *Boundary behavior of $\bar{\partial}$ on weakly pseudoconvex manifolds of dimension two*, J. Differ. Geom. 6 (1972), 523-542.
- [K74] J.J. Kohn, *Subellipticity on pseudo-convex domains with isolated degeneracies*, Proc. Nat. Acad. Sci. U.S.A. 71 (1974), 2912–2914.
- [K79] J.J. Kohn, *Subellipticity of the $\bar{\partial}$ -Neumann problem on pseudo-convex domains: sufficient conditions*, Acta Math. 142 (1979), 79-122.
- [KN] J.J. Kohn and L. Nirenberg, *Non-coercive boundary value problems*, comm. pure Appl. Math. 18 (1965), 443-492.
- [M92] J.D. McNeal, *Convex domains of finite type*, J. Funct. Anal. 108 (1992), no. 2, 361–373
- [M02] J.D. McNeal, *Uniform subelliptic estimates on scaled convex domains of finite type*, Proc. Amer. Math. Soc. 130 (2002), no. 1, 39–47.
- [Ni07] A.C. Nicoara, *Equivalence of types and Catlin boundary systems*, arXiv:0711.0429.
- [Ni08] A.C. Nicoara, *The Kohn Algorithm on Denjoy-Carleman Classes*, arXiv:0806.1917.
- [Siu] Y.-T. Siu, *Effective Termination of Kohn’s Algorithm for Subelliptic Multipliers*. arXiv:0706.4113.